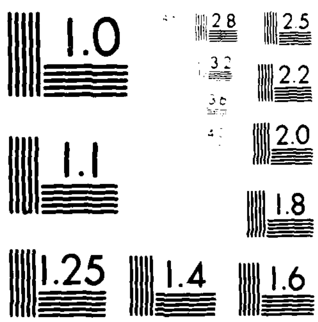

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Weak convergence of a Bayesian nonparametric estimator of the survival function under progressive censoring

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Abstract

A nonparametric Bayesian estimator $\hat{F}$ of the survival function F constructed from time-sequential progressively censored observations is found to subsume several estimators of F utilized in practice. Weak convergence of $\hat{F}$ is developed and the limiting process is found to coincide with that obtained when complete response profiles of the sample are available, leading to suitable application of $\hat{F}$ with consequent reductions in costs and time and without loss of asymptotic accuracy.

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§1. INTRODUCTION/SUMMARY.

The basic formulation of the problem proposed here has been introduced and studied from a Bayesian viewpoint in [5]. It involves consideration of a set of independent random lifetimes $X_1,\ldots,X_n$ which may be deterred from complete observation due to censoring on the right by another set of independent variables $Y_1,\ldots,Y_n$. Thus the observable set of data is $\{(Z_i,\delta_i): 1 \leq i \leq n\}$ where for each $1 \leq i \leq n$,

$$(1.1) \qquad Z_i = X_i \wedge Y_i \text{ and } \delta_i = 0 \text{ or } 1 \text{ according as } X_i > Y_i \text{ or not.}$$

This is the commonly called random censorship model which has received considerable attention among researchers in the past several years. However, as has been observed in [5] there is a broad class of experiments pertaining to clinical trials and reliability in which the Z_i are observed sequentially and cost and/or time considerations often entail termination of experimentation before all Z_i have been observed. For example a study may be curtailed at the $k = k(n)$th smallest order statistic $Z_{(k)}$, $1 \leq k \leq n$ and then, in effect, the investigator has at his disposal only the data

$$(1.2) \qquad \{(Z_{(i)},\delta_i^*): 1 \leq i \leq k_{(n)}\} \text{ with } \delta_i^* = 1 \text{ or } 0 \text{ according as } Z_{(i)}$$

is a true lifetime or censoring time.

Statistical procedures based on data of the type (1.2) are referred to as progressively censored schemes (Sen et. al (1973, 1978)). Thus instead of prolonged observational periods until responses from the entire sample have been recorded, experimentation may be ceased at an appropriate intermediate stage with an attendant desirable reduction in costs and time.

In [5] the Bayes estimator $\hat{F}_{k(n)}$ of the survival function

$$F(t) = P[X_1 > t|F], \ t \geq 0 \tag{1.3}$$

has been obtained from the data (1.2), under squared-error loss when F follows a Dirichlet process prior. We have noted there that $\hat{F}_{k(n)}$ yields as special cases, the Ferguson (1973) estimator of F whenever all $X_1,\ldots,X_n$ are observable; the usual empirical survival function of $X_1,\ldots,X_n$ and the estimators given by Kaplan and Meier (1958) and Susarla and Van Ryzin (1976). Our objective in this paper is the investigation of the weak convergence of $\hat{F}_{k(n)}$.

We shall show here that the process $\{n^{\frac{1}{2}}(\hat{F}_{k(n)}(t)-F(t)):$ $0 \leq t \leq T\}$, $T < \infty$ has exactly the same asymptotic properties as the process $\{n^{\frac{1}{2}}(\hat{F}_n(t)-F(t)):$ $0 \leq t \leq T\}$ provided $\liminf n^{-1}k(n) > 1-P[Z_1 > T|F]$ and the sequences $\{X_i:$ $i \geq 1\}$, $\{Y_i,$ $i \geq 1\}$ being each independent and identically distributed (iid) and independent of each other. In view of this it is not unreasonable in practice to terminate experimentation at the $k(n)$th stage when $n^{-1}k(n)$ is close to unity. For example if $k(n) = n-c \log n$ with $c > 0$ large, substantial savings in time and cost can be afforded without sacrifice of asymptotic accuracy yielding a very cost-effective procedure especially when per unit observational costs are prohibitively high.

The substantive material of the paper appears in the next two sections. Section 2 introduces some notation and assumptions together with a discussion of our estimator and special cases of it. The proof of the weak convergence is detailed in Section 3 and we make a few concluding remarks in the final section.

§2. Preliminaries

Consider $n(\geq 1)$ units under surveillance for which we record either the time to decrement (survival time) X or its competing censoring time Y upto and including the k_n-th response, $k_n \in \{1,\ldots,n\}$. Suppose the survival distribution F of X is a Dirichlet process with parameter measure α on the Borel subsets of the positive real line $\mathbb{R}^+ = (0,\infty)$ and, given F, the survival times $X_1,\ldots,X_n$ are iid with distribution function (d.f.) $(1-F)$. Furthermore, when the corresponding censoring times $Y_1,\ldots,Y_n$ are iid with continuous right df G on $(0,\infty)$ and independent of $(F,X_1,\ldots,X_n)$, we have demonstrated in [5] that under weighted squared-error loss the Bayes estimator $\hat{F}_{k_n}$ of the survival curve F of (1.3) can be written

$$\hat{F}_{k_n}(t) = B_n(t)W_n(t) \tag{2.1}$$

where

$$B_n(t) = \frac{\alpha(t) + N_n^+(t) + (n-k_n)[Z_{(k_n)} > t]}{\alpha(0) + n}, \tag{2.2}$$

$$W_n(t) = \prod_{j=1}^{k_n} \left\{\frac{\alpha(Z_{(j)}) + N_n^+(Z_{(j)}) + (n-k_n) + 1}{\alpha(Z_{(j)}) + N_n^+(Z_{(j)}) + (n-k_n)}\right\}^{[Z_{(j)} \leq t,\delta_j^* = 0]} \tag{2.3}$$

$$\cdot\left\{\frac{\alpha(Z_{(k_n)}) + (n-k_n)}{\alpha(Z_{(k_n)})}\right\}^{[Z_{(k_n)} \leq t]}$$

with the abbreviation $\alpha(u) = \alpha(u,\infty)$ and

$$N_n^+(t) = \sum_{j=1}^{k_n} [Z_{(j)} > t] \tag{2.4}$$

with $[A]$ denoting the indicator of the set A. We shall assume throughout that $\alpha(T) > 0$ and $G(T) > 0$ where $0 < T < \infty$ is fixed.

The estimator (2.1) may be regarded as a natural extension of an estimator of $F(t)$ obtained by Susarla and Van Ryzin (1976) when the complete data set $\{(Z_i, \delta_i): 1 \leq i \leq n\}$ is available. Indeed (2.1) yields this estimator by setting $k_n = n$. On the other hand if there are no observed censoring times in the data $\{(Z_{(i)}, \delta_i^*): 1 \leq i \leq k_n\}$ (2.1) reduces to

$$(2.5) \quad \hat{F}(t) = \left\{ \frac{\alpha(t) + N_n^+(t) + (n-k_n)[Z_{(k_n)} > t]}{\alpha(0) + n} \right\} \left\{ \frac{\alpha(Z_{(k_n)}) + (n-k_n)}{\alpha(Z_{(k_n)})} \right\}^{[Z_{(k_n)} \leq t]}$$

which in turn we may view as a generalization of Ferguson's (1973) estimator $\frac{\alpha(t) + N_n^+(t)}{\alpha(0) + n}$, when censoring is absent and $k_n = n$. The Ferguson estimator reduces to the empirical survival function (of $X_1, \ldots, X_n$) in the limit as $\alpha(0) \to 0$, while (2.1) for $t \leq Z_{(k_n)}$ and the Susarla-Van Ryzin estimator (that is (2.1) with $k_n = n$) yields the product-limit estimator of Kaplan and Meier (1958), which itself reduces to the empirical survival function in the absence of censoring. We thus notice that several estimators of the survival curve that have found favor in a wide variety of practical situations have a common (hitherto unborn) progenitor - the estimator $\hat{F}_{k_n}$ of (2.1).

§3. Weak convergence of $\hat{F}_{k_n}$.

The strategy here is to express $n^{\frac{1}{2}}(\hat{F}_{k_n}-F)$ in a more tractable form in which its behavior will be made transparent from the separate behavior of B_n and W_n. We begin with B_n. Define

$$(3.1) \qquad H_n(t) = n^{-1}\sum_{j=1}^{n}[Z_j > t],\ H(t) = P[Z_1 > t|F] = F(t)G(t).$$

Then from (2.4) $N_n^+(t) = \{nH_n(t)-(n-k_n)\}[Z_{(k_n)} > t]$ and in conjunction with (2.2) this yields

$$(3.2) \quad B_n(t)-H(t) = -\{\alpha(0) + n\}^{-1}\{n(H_n(t)-H(t))-nH_n(t)[Z_{(k_n)} \le t] -\alpha(0,t)\}.$$

Writing $\|\cdot\|_T$ for the sup-norm on $(0,T]$ we get

$$(3.3) \qquad \|B_n-H\|_T \le \{\alpha(0) + n\}^{-1}\{n\|H_n-H\|_T + n[Z_{(k_n)} \le T] + \alpha(0,T)\}.$$

In the sequel we shall repeatedly use the fact that $\|H_n-H\|_T = o_p(n^{-\beta})$ if $\beta < \frac{1}{2}$. Thus with $\beta < \frac{1}{2}$ we will also have from (3.3)

$$(3.4) \qquad \|B_n-H\|_T = o_p(n^{-\beta})$$

once we demonstrate

$$(3.5) \qquad P[Z_{(k_n)} \le T|F] = o_p(n^{-\beta}),\ \text{whatever}\ \beta.$$

Now assume that the sequence of stopping numbers $\{k_n:\ n \ge 1\}$ satisfies

$$(3.6) \qquad \gamma \equiv \liminf n^{-1}k_n > 1-H(T),\ \text{with}\ \gamma \in (0,1].$$

Then there exists an integer n_0 (depending on γ and $H(T)$) such that $n^{-1}k_n > \frac{1}{2}\{\gamma + 1-H(T)\}$ whenever $n \geq n_0$. Hence with $n \geq n_0$

$$P[Z_{(k_n)} \leq T|F] \leq P[(1-H_n(T))-(1-H(T)) \geq \tfrac{1}{2}(\gamma-(1-H(T)))|F].$$

We bound this right hand side probability by using the Bernstein inequality (see Hoeffding (1963)), whence

(3.7) $$P[Z_{(k_n)} \leq T|F] \leq \exp(-\tfrac{1}{2}n\{\gamma-(1-H(T))\}^2)$$

and now (3.5) obtains, whatever β.

To handle W_n we have to work much harder. (In what follows we use f^{-1} to denote the reciprocal of the function f rather than its inverse.) First write (2.3) in the form

(3.8) $$W_n = W_{n,1}W_{n,2}, \text{ where } W_{n,2}(t) = \left\{\frac{\alpha(Z_{(k_n)}) + (n-k_n)}{\alpha(Z_{(k_n)})}\right\}^{[Z_{(k_n)} \leq t]}.$$

Then since $\alpha(u)$ is nonincreasing in u, we obtain

$$\| \ln W_{n,2}\|_T \leq [Z_{(k_n)} \leq T]\ln \{1 + (n-k_n)\alpha^{-1}(Z_{(k_n)})\}$$
$$\leq [Z_{(k_n)} \leq T]n\alpha^{-1}(T),$$

which in turn yields, in view of (3.5)

(3.9) $$\| \ln W_{n,2}\|_T = o_p(n^{-\beta}), \text{ whatever } \beta.$$

Now introduce

(3.10) $$\tilde{H}_n(t) = n^{-1}\sum_{j=1}^{n}[Z_j \leq t,\delta_j = 0]; \quad \tilde{H}(t) = P[Z_1 \leq t,\delta_1 = 0|F].$$

Then from (2.3), (3.1) and (3.8) we can express

$$(3.11) \qquad \ln W_{n,1}(t) = n \int_0^{Z_{(k_n)} \wedge t} \ln\{1 + (\alpha(x) + nH_n(x))^{-1}\} d\tilde{H}_n(x).$$

On utilizing the expansion $\ln(1 + x) = x + x^2 \sum_{j=0}^{\infty} (-1)^{j-1} \frac{x^j}{j+2}$,
$0 < x \leq 1$ in the integrand we rewrite (3.1) as

$$(3.12) \qquad \ln W_{n,1}(t) = K_{n,1}(t) + K_{n,2}(t).$$

Since both $\alpha(x)$, $H_n(x)$ are nonincreasing in x, we get on simplification

$$\|K_{n,2}\|_T \leq \frac{n}{\{\alpha(T)+nH_n(T)\}^2} \; \frac{\tilde{H}_n(T)}{\{\alpha(T)+nH_n(T)-1\}},$$

which in conjunction with the fact that $\tilde{H}_n(T) = \tilde{H}(T) + o_p(n^{-\beta})$ (with $\beta < \frac{1}{2}$) yields

$$(3.13) \qquad \|K_{n,2}\|_T = o_p(n^{-\beta}), \text{ whenever } \beta < \tfrac{1}{2}.$$

Now observe that $\tilde{H}$ of (3.10) can also be written $\int_0^t F(x)\{-dG(x)\}$ and therefore, whenever $H(t) > 0$, $\frac{d}{dt}\{\ln G^{-1}(t)\} = H^{-1}(t)d\tilde{H}(t)$. Hence from (3.12)

$$\begin{aligned}
(3.14) \quad K_{n,1}(t) - \ln G^{-1}(t) &= n \int_0^{Z_{(k_n)} \wedge t} \{\alpha(x) + nH_n(x)\}^{-1} d\tilde{H}_n(x) - \int_0^t H^{-1}(x) d\tilde{H}(x) \\
&= \int_0^t [Z_{(k_n)} \geq x](n\{\alpha(x) + nH_n(x)\}^{-1} - H^{-1}(x)) d\tilde{H}_n(x) \\
&\quad + \int_0^t H^{-1}(x) d\{\tilde{H}_n(x) - \tilde{H}(x)\} - \int_0^t [Z_{(k_n)} < x] H^{-1}(x) d\tilde{H}_n(x) \\
&\equiv K_{n,3}(t) + K_{n,4}(t) + K_{n,5}(t).
\end{aligned}$$

We first dispose of $K_{n,4}$ and $K_{n,5}$. By noting that $H^{-1}(x)$ is nondecreasing in x and performing an integration-by-parts, we easily obtain

$$\|K_{n,4}\|_T \leq 2\, H^{-1}(T)\|\tilde{H}_n - \tilde{H}\|_T$$

and in view of the fact that $\|\tilde{H}_n - \tilde{H}\|_T = o_p(n^{-\beta})$, $\beta < \frac{1}{2}$

(3.15) $$\|K_{n,4}\|_T = o_p(n^{-\beta}), \text{ whenever } \beta < \tfrac{1}{2}.$$

Also $\|K_{n,5}\|_T \leq H^{-1}(T)\tilde{H}_n(T)I[Z_{(k_n)} \leq T]$, which gives, using (3.5),

(3.16) $$\|K_{n,5}\|_T = o_p(n^{-\beta}), \text{ whatever } \beta.$$

Finally, $\|K_{n,3}\|_T \leq \int_0^T |n\{\alpha(x) + nH_n(x)\}^{-1} - H^{-1}(x)\}|d\tilde{H}_n(x)$

$$\leq \{(\alpha(T) + nH_n(T))H(T)\}^{-1}\tilde{H}_n(T)\{\alpha(0) + n\|H_n - H\|_T\}$$

which leads to

(3.17) $$\|K_{n,3}\|_T = o_p(n^{-\beta}), \text{ whenever } \beta < \tfrac{1}{2}.$$

Thus collecting our results (3.9) and (3.13) through (3.17), we finally obtain for (3.8)

(3.18) $$\|\ell n\, W_n - \ell n\, G^{-1}\|_T = o_p(n^{-\beta}), \text{ whenever } \beta < \tfrac{1}{2}$$

We are now in position to express $n^{\frac{1}{2}}(\hat{F}_{k_n} - F)$ in a governable form. Write

(3.19) $$(\hat{F}_{k_n} - F) = B_n W_n - HG^{-1}$$

$$= (B_n - H)G^{-1} + (W_n - G^{-1})B_n.$$

On employing the expansion $e^x = 1 + x + \frac{x^2}{2}e^c$, where c lies between 0 and x, we can write

$W_n - G^{-1} = G^{-1}(\ln W_n - \ln G^{-1}) + \frac{1}{2} G^{-1} e^c (\ln W_n - \ln G^{-1})^2$, and so from (3.19) we have

$$\| n^{\frac{1}{2}}(\hat{F}_k - F) - n^{\frac{1}{2}}(B_n - H)G^{-1} - n^{\frac{1}{2}}(\ln W_n - \ln G^{-1})H\, G^{-1}\|_T$$

$$\leq \tfrac{1}{2}\, n^{\frac{1}{2}} \|\ln W_n - \ln G^{-1}\|_T^2 + n^{\frac{1}{2}} \|B_n - H\|_T \|\ln W_n - \ln G^{-1}\|_T G^{-1}(T)$$

which in view of (3.4) and (3.18) is $o_p(n^{-\beta})$, for some $\beta < \frac{1}{2}$. This reduces consideration of weak convergence of $n^{\frac{1}{2}}(\hat{F}_{k_n} - F)$ to that of

$$(3.20) \qquad n^{\frac{1}{2}}(B_n - H)G^{-1} + n^{\frac{1}{2}}(\ln W_n - \ln G^{-1})HG^{-1}$$

In (3.20) we can further replace W_n by $W_{n,1}$ in view of (3.9). The final simiplifcation is obtained by writing (see 3.14),

$$(3.21) \qquad \ln W_{n\,1}(t) - \ln G^{-1}(t) = \{\int_0^t H_n^{-1}(x)d\tilde{H}_n(x) - \int_0^t H^{-1}(x)d\tilde{H}(x)\} + C_n(t)$$

where $\displaystyle C_n(t) = -\int_0^t \frac{[Z_{(k_n)} \geq x]\alpha(x)}{H_n(x)\{\alpha(x) + nH_n(x)\}} d\tilde{H}_n(x) - \int_0^t \frac{[Z_{(k_n)} < x]}{H_n(x)} d\tilde{H}_n(x)$.

But $\displaystyle \|C_n\|_T \leq \frac{\alpha(0)\tilde{H}_n(T)}{H_n(T)\{\alpha(T) + nH_n(T)\}} + [Z_{(k_n)} \leq T] \frac{\tilde{H}_n(T)}{H_n(T)}$, and again

$\|C_n\|_T = o_p(n^{-\beta})$, whenever $\beta < \frac{1}{2}$. Thus the weak convergence of $\{n^{\frac{1}{2}}(\hat{F}_{k_n}(t) - F(t)):\ t \in [0,T]\}$ is tantamount to that of

$$(3.22) \qquad n^{\frac{1}{2}}\{B_n(t) - H(t)\}G^{-1}(t) + n^{\frac{1}{2}}\{\int_0^t H_n^{-1} d\tilde{H}_n - \int_0^t H^{-1} d\tilde{H}\}$$

which is precisely the final reduction obtained by Susarla and Van Ryzin (1978). We must however emphasize here that the interval endpoint T is restricted through (3.6). As noted earlier the estimator F_{k_n}

reduces to that studied by Susarla and Van Ryzin (1976, 1978) when $k_n = n$, in which situation (3.6) degenerates to the condition $H(T) > 0$ that they imposed.

The analysis of the process (3.22) has been dealt with in Susarla and Van Ryzin (1978) and Breslow and Crowley (1974). We therefore state our results for $\hat{F}_{k_n}$ in

Theorem: Suppose $\hat{F}_{k_n}$ is defined by (2.1)-(2.4) and the sequence $\{k_n: n \geq 1\}$ satisfies (3.6). Assume $F(t)$, $G(t)$ continuous on $[0,T]$. Then the process $\{n^{\frac{1}{2}}(\hat{F}_{k_n}(t)-F(t)): t \in [0,T]\}$ converges weakly *to a Gaussian process*

$$U = -G^{-1}P + \int_0^{\cdot} H^{-2}Pd\tilde{H} + H^{-1}Q + \int_0^{\cdot} H^{-2}QdH$$

where P,Q are themselves mean zero Gaussian processes with a covariance structure given for $s \leq t$, by

$$\begin{aligned}
\mathrm{Cov}(P(s), P(t)) &= H(t)(1-H(s)) \\
\mathrm{Cov}(Q(s), Q(t)) &= \tilde{H}(s)(1-\tilde{H}(t)) \\
\mathrm{Cov}(P(s), Q(t)) &= \tilde{H}(s)-\tilde{H}(t)(1-H(s)) \\
\mathrm{Cov}(Q(s), P(t)) &= \tilde{H}(s)H(t).
\end{aligned}$$

Indeed U has mean zero and covariance given for $s \leq t$ by

$$\mathrm{Cov}(U(s), U(t)) = -F(s)F(t) \int_0^s H^{-1}F^{-1}dF.$$

§4. Concluding Remarks.

In this paper we have assumed the censoring variables $Y_1,\ldots,Y_n$ to be iid. Under appropriate conditions our results can be extended to the case where the Y_i's have different distributions. In addition to the weak convergence result for $\{F_{k_n}: n \geq 1\}$ one can demonstrate the uniform strong consistency and uniform mean-square consistency of $\{F_{k_n}: n \geq 1\}$ as a estimator of F on $[0,T]$.

Finally confidence bands for $\{F(t): t \in [0,T]\}$ can be calculated in terms of $\{F_{k_n}(t): t \in [0,T]\}$ using the distribution of the limiting process $\{U(t): t \in [0,T]\}$ of our Theorem. Note that the process $\{n^{\frac{1}{2}}\frac{(\hat{F}_{k_n}(t) - F(t))}{F(t)}: t \in [0,T]\}$ converges weakly to a time-transformed Brownian motion.

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